Singularity spectrum of the velocity increment in isotropic turbulence

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A reasonable picture of the singularity spectrum $d(h)$ of the longitudinal velocity increment presumed by Frisch and Parisi is given. The Kolmogorov refined similarity hypothesis is found to play a vital role in constructing the $d(h)$ spectrum in the entire range of h from the knowledge of the $f(\alpha)$ spectrum for the three-dimensional dissipation field of turbulence. The feature of $d(h)$ obtained for the lognormal model is discussed in comparison with other models. Although the lognormal model itself is now known not to be the best one, it is typical in that it is analytically treated without a numerical process and it has a universal tendency, at least, in the right branch of $d(h)$.

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Since Frisch and Parisi [1], it has been expected that longitudinal velocity increment $|\Delta u_r|$ across distance r in the inertial range in isotropic turbulence is self-similar like to $\sim r^h$ with stochastic singularity strength h, and that the interwoven spatial set supporting a value of h makes a fractal with the dimension $d(h)$, which is often called the singularity spectrum of Δu_r. Although wavelet analysis [2] was anticipated as a powerful tool for directly measuring the spatial distribution of h and hence $d(h)$, the perfect form of $d(h)$ does not yet seem to have been fixed. To be sure, Muzy, Bacry, and Arneodo [3] reached a conclusion about a picture of $d(h)$ which was consistent with a transformed form of the $f(\alpha)$ spectrum for the dissipation field of turbulence in a one-dimensional (1D) cut by Meneveau and Sreenivasan [4]. However, there still remains the problem that if we find the maximum of h ($h_{\max}$, apparently close to 0.6) that way, we can never find (a finite amount of the probability of) $|\Delta u_r|\cong 0$ for a finite value of r in the inertial range since h cannot go to ∞ (and since the Hölder constant is never presumed to be zero). This is in contradiction to fact; indeed, it is well known that the probability density function (PDF) of Δu_r is finite and smooth at the origin [5,6], suggesting that $h_{\max}$ should be ∞. Therefore, the behavior of at least the right branch of $d(h)$ presented there is irrational.

This paper presents a reasonable picture of $d(h)$ over a whole region of h on the basis of the Kolmogorov refined similarity hypothesis (RSH) [7] and the 3D $f(\alpha)$ spectrum given by the lognormal model [7,8]. The lognormal model prescribes that the random multiplier in the cascade process of energy dissipation distributes lognormally. It is well known that this model has flaws [9–11]: it violates the Novikov constraint and the qth-order intermittency exponents predicted by the model increase too rapidly with $|q|$ [12]. However, we notice here that it is probably the only model possible to give the analytical expression of $d(h)$, and that it is still reliable in predicting low-order intermittency exponents [13,14]. Therefore it is meaningful to give the $d(h)$ of the lognormal model first, and compare it with that of other improved models next. This will clarify which part of the $d(h)$ of the lognormal model should be unreliable and, conversely, which other part be reliable.

First we note a delicate statistical difference between Δu_r and $(r\varepsilon_r)^{1/3}$ (ε_r is the dissipation rate averaged over a sphere with the diameter of r), from which the difference of $f(\alpha)$ and $d(h)$ will arise. The Kolmogorov RSH implies that

$$\Delta u_r = v(r\varepsilon_r)^{1/3} , \tag{1}$$

where v is a universal stochastic variable independent of r and ε_r in the inertial range of r. In accordance with recent experimental evidence [15,16], the PDF of v, $P(v)$, is nearly Gaussian with the zero mean. Thus we can express the PDF of Δu_r in terms of a production sum of two independent PDF's of v and $x\equiv(r\varepsilon_r)^{1/3}$ as

$$p_3(\Delta u_r) = \int P(\Delta u_r/x)/x p_2(x) dx . \tag{2}$$

This is a statistical expression for the Kolmogorov RSH [although he did not specify $P(v)$ to be Gaussian]. The exact form of $p_2(x)$ is already known for many models of isotropic turbulence [17].

Consider the pth-order moments of both sides of (1), and we have

$$\int |\Delta u_r|^p p_3(\Delta u_r) d\Delta u_r = \int\int |v|^p x^p P(v) p_2(x) dv\, dx \propto r^{p/3-\mu(p/3)} , \tag{3}$$

where $\mu(q)$ is the qth-order intermittency exponent, which implies $\langle(\varepsilon_r/\varepsilon_l)^q\rangle = (r/l)^{-\mu(q)}$ for any ratio of $r/l<1$, when a domain of scale r is included in a domain of l; the concept of $\mu(q)$ was introduced by Novikov [10], with the terminology of scale similarity of dissipation measure. Usually, the exponent of the right-hand side is called $\zeta(p)$; that is

$$\zeta(p) = p/3 - \mu(p/3) . \tag{4}$$

Here p must be limited to be greater than -1. Some negative value of p can be allowed since $p_3(\Delta u_r)$ is finite and continuous at $\Delta u_r=0$. It is well known that $\mu(q)$ is related to the $f(\alpha)$ spectrum through the Legendre transformation:

$$-\mu(q)=3-f(\alpha)+q(\alpha-1)\,,$$
$$q=f'(\alpha)\,. \qquad (5)$$

[The usual expression [18] in terms of generalized dimensions $D(q)$ is regenerated by using the relation $(D(q)-3)(q-1)=-\mu(q)$.] In a similar way, we may have the Legendre transformation [1] which relates $\zeta(p)$ to the $d(h)$ spectrum,

$$\zeta(p)=3-d(h)+ph\,,$$
$$p=d'(h)\,. \qquad (6)$$

Since Eqs. (4), (5), and (6) are isomorphic,

$$d(h)=f(3h) \qquad (7)$$

is obtained, so far as $p=d'(h)>-1$.

The analytical form of $f(\alpha)$ can be obtained easily for the lognormal model. First, for the lognormal model, we know that

$$p_2(x)=3/[(2\pi)^{1/2}sx]\times\exp[-(3\ln x-\ln r-\mu\ln r/2)^2/(2s^2)]\,, \qquad (8)$$

with $s^2=-\mu\ln r$ [17]. (μ is the second-order intermittency exponent μ_2.) Here x and r are normalized by $(L\varepsilon_L)^{1/3}$ and L. (L is the main scale.) Second, we transform the variable x to α by the self-similar relation $x^3=r^\alpha$ (equivalent to $\varepsilon_r=r^{\alpha-1}$). Then we have the PDF of α:

$$p_4(\alpha;r)=[|\ln r|/(2\pi\mu)]^{1/2}r^{(\alpha-1-\mu/2)^2/2\mu}\,. \qquad (9)$$

This means that

$$f(\alpha)=-(\alpha-1-\mu/2)^2/2\mu+3\,, \qquad (10)$$

in accordance with the general formula for the PDF of exponent α of a scale-similar measure [13,19,20]

$$P(\alpha;r)=r^{3-f(\alpha)}[|f''(\alpha)\ln r|]/(2\pi)]^{1/2} \quad \text{as } r\to 0\,. \qquad (11)$$

Thus the form of $d(h)$ can be known through (7) until $p=d'(h)$ arrives at -1, that means $q=-\frac{1}{3}$. It is easy to derive from (7) and (10) that the limiting point is located at $((1+5\mu/6)/3, 3-\mu/18)$ in the h-d plane. Beyond this point (7) is obviously illusive. The next important problem is, thus, how to grasp the behavior of $d(h)$ for a larger h toward ∞.

We consider (2) again with (8), changing the variable $|\Delta u_r|$ to r^h and x to $r^{\alpha/3}$. Then we obtain the PDF of h

$$p_5(h;r)=r^h(|\ln r|)^{3/2}(\pi\sigma\mu^{1/2})\times\int r^{-\alpha/3}\exp[-r^{2(h-\alpha/3)}/(2\sigma)^2+(\alpha-1-\mu/2)^2\ln r/(2\mu)]d\alpha\,. \qquad (12)$$

Here σ is the standard deviation of a Gaussian function $P(v)$ which solely depends on how to normalize Δu_r. For a very small r the integrand in (12) almost vanishes for $\alpha>3h$, so that the integral may be written as $\int^{3h}r^{-\alpha/3}\exp[(\alpha-1-\mu/2)^2\ln r/(2\mu)]d\alpha$. This can be estimated by the saddle point method on the condition that the saddle point α_c should be less than $3h$. Since $\alpha_c=1+5\mu/6$, we have

$$p_5(h;r)=(2/\pi)^{1/2}/\sigma r^h|\ln r|r^{-1/3-2\mu/9}\sim r^{3-d(h)} \qquad (13)$$

only for $h>\alpha_c/3=(1+5\mu/6)/3$, and, hence,

$$d(h)=3-h+\tfrac{1}{3}+2\mu/9 \qquad (14)$$

there. Since $d(\alpha_c/3)=3-\mu/18$, this straight line with $d'(h)=-1$ smoothly connects to the $d(h)$ curve prescribed in the preceding paragraph just at its limiting point.

Thus it is evident that the $d(h)$ curve for the lognormal model of isotropic turbulence is composed of the parabolic curve given by (7) and (10) as the left branch, and the straight line tangent to that with the gradient -1 as the right branch. This is the exact implication of (12), which comes from (2). On this $d(h)$ curve, naturally we have no possibility of having $p<-1$, while h itself can be infinitely large. It is the right branch of $d(h)$ smoothly grafted by (13) that governs the behavior of $p_3(\Delta u_r)$ near $\Delta u_r=0$. The same thing is true for many other models. This feature of the $d(h)$ spectrum was already verified for the 3D binomial Cantor set model [13] partially with a numerical process [21], although the left branch is no longer parabolic in this case. Since the mathematical structure of this model is similar to all binomial Cantor set models [13], including the p model [4] and the random β model [22,23], the same feature of the $d(h)$ spectrum will be easily identified for such models. Most of the right branch may be called the "regularity spectrum" of the velocity increment, since $h\geq 1$ indicates no singularity.

The real $d(h)$ for isotropic turbulence may be conjectured from the 3D $f(\alpha)$ spectrum obtained by a direct numerical simulation (DNS), which treats a decaying turbulence [12]. Another recent DNS (treating a forced turbulence) by Chen supports very closely this result for a wide range of Taylor-scale Reynolds numbers [24]. [We note that his $f(\alpha)$ is Reynolds number independent.] According to these, we have no negative h since the lower limit of α is positive, where f seems to accumulate to a finite value as $q\to\infty$. Therefore the parabola as the left branch given by the lognormal model is not suitable particularly for small α and then h. However, as is well known, the lognormal model is a very good approximation for small $|q|$; in fact, there is no difference seen between the DNS result and the lognormal model in the neighborhood of the summit of $f(\alpha)$, which corresponds to small $|q|$ [12]. Then we can consider that the right branch of $d(h)$ that the lognormal model brings forth is reliable, since the limiting point $h=\alpha_c/3$ is very near the summit point, $h=(1+\mu/2)/3$. In other words, the tangent lines with $d'(h)=-1$ that all reasonable models bring forth must nearly collapse. The result of the 3D binomial Cantor set model is satisfactory on this point, and also the behavior of the left branch of its $d(h)$ improves that of the lognormal model remarkably [21].

It is not surprising that $d(h)$ is negative for large h. The true sense of negative fractal dimension (beyond the Hausdorff dimension) was often explained by Mandelbrot [25], and even an experimental pursuit of the negative part of $f(\alpha)$ was made by Chhabra and Sreenivasan [26]. It is sufficient here to understand that the probability of finding the set supporting h in space is extremely rare if $d(h)$ is negative; the rarer, the more negative. Recently, Mandelbrot [27] proposed a similar straight-line matching with the parabola of the lognormal model as a new model. But he treated $f(\alpha)$ not $d(h)$, and the matching was made on the left branch of $f(\alpha)$. Therefore the physical content of his model is quite different from that of the present problem.

Finally we add that the reliability of formulating $p_3(\Delta u_r)$ essentially based on RSH was already acknowledged in comparison with DNS [17,28] and experiment [29]. This fact indirectly shows that the $d(h)$ spectrum just considered for $|\Delta u_r|=r^h$ on the RSH basis has sufficient reality.

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